Multi-Item capacitated lot-Sizing problem with setup times and shortage costs : Polyhedral results

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Abstract

We address a multi-item capacitated lot-sizing problem with setup times and shortage costs that arises in real-world production planning problems. Demand cannot be backlogged, but can be totally or partially lost. The problem can be modelled as a mixed integer program and it is NP-hard. In this paper, we propose some classes of valid inequalities based on a generalization of Miller *et al.* [17] results. We study the polyhedral structure of the convex hull of this model which help us to prove that these inequalities induce facets of the convex hull under certain conditions.
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Introduction

The Multi-item Capacitated Lot-Sizing Problem with with Setup times and Shortage costs (MCLSSP) is a production planning problem in which there is a time-varying demand for a set of N items denoted $\mathcal{I} = \{1, 2, \cdots, N\}$ over T periods. The production should satisfy a restricted capacity and must take into account a set of additional constraints. Indeed, launching the production of an item i at a given period t for a demand requirement d_{it} involves a variable capacity v_{it} and a fixed consumption of resource f_{it} usually called setup time. The total available capacity at period t is c_t. The production should also satisfy lot-sizing constraints. For each period t, an inventory cost γ_{it} is attached to each item i as well as a variable unit production cost α_{it} and a setup cost β_{it}. The problem has the distinctive feature of allowing requirement shortages because we deal with problems with tight capacities. Indeed, when we are in lack of capacity to produce the total demand, we try to spread the capacity among the items by minimizing the total amount of demand shortages. Thus, we introduce in the model a unit cost parameter φ_{it} for item i at period t for the requirement not met regarding the demand. These costs should be viewed as penalty costs and their values are very high in comparison with other cost components.
To try to meet the demand for an item i at period t, we could anticipate the production over some periods of time. Therefore, σ_{it} denote the last period at which an item i produced at period t can be consumed. The problem MCLSSP is to find a production planning that minimizes the demand shortage, the setup, the inventory and the production costs.

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Florian et al. [10] and Bitran and Yanasse [4] have shown that the single-item capacitated lot-sizing problem is NP-hard, even for many special cases. Chen and Thizy [6] have proved that multi-item capacitated lot-sizing problem (MCLSP) with setup times is strongly NP-hard. The problem of finding a feasible solution to the MCLSP problem with setup times is NP-Complete (Bin packing is a special case of MCLSP with setup times, see [18]).

Since the seminal papers by Wagner and Within [24] and Manne [14] in the late 1950s, a lot of research has been done on lot-sizing problems. The single-item problem has been given special interest for its relative simplicity and for its importance as a sub-problem of some more complex lot-sizing problems. For a complete review, the reader can refer to [5].

Although production planning models involving multiple items, restrictive capacities and significant setup times occur frequently in industrial situations and have often been studied in the literature, obtaining optimal and sometimes even feasible solutions remains challenging. Trigeiro *et al.* [23] were among the firsts to try to solve such models. They proposed a lagrangean relaxation based heuristic to solve the single-machine, multi-item, capacitated lot-sizing problem with setup times to obtain near-optimal solutions. Since the lagrangean solutions were not always feasible, they used a production smoothing heuristic in order to obtain feasible production plans. However, we can notice that for all the instances with tight capacities, they were not able to find feasible solutions.

Pochet and Wolsey [21], Belvaux and Wolsey [3] and Leung *et al.* [12] proposed exact methods to solve multi-item capacitated lot-sizing problems by strengthening the LP formulation with valid inequalities and then using a MIP solver. Barany *et al.* [2] have defined some inequalities for the uncapacitated lot-sizing problem. Miller *et al.* [17] have studied the polyhedral structure of some capacitated production planning problems with setup times. We can also mention the work of Marchand and Wolsey [15] for the 0-1 knapsack problem which appear as a relaxation of a number of structured MIP problems such as the MCLSP problem.

There are few works addressing lot-sizing problems with shortage costs. Recently, Sandbothe and Thompson [22] addressed a single-item uncapacitated lot-sizing problem with shortage costs. The authors proposed a $O(T^3)$ forward dynamic programming algorithm to solve the problem optimally. Aksen *et al.* [1] proposed a dynamic programming algorithm to solve the same problem optimally in $O(T^2)$ time. Loparic *et al.* [13] proposed valid inequalities for the single-item uncapacitated lot-sizing problem with sales and lower bounds on stock variables. The authors considered sales instead of fixed demands and shortages. In our knowledge, we are the first to combine setup times and shortage cost for a multi-item capacitated lot-sizing problem.

In the literature two methods are presented for dealing with problems where demand cannot be met in every period. Dixon *et al.* [9] deal with lack of capacity by considering overtimes. The capacity constraint is expanded by making extra capacity available at a cost. The multi-item capacitated lot-sizing problem with setup times and overtime decisions is investigated by Diaby *et al.* [8], Özdamar and Birbil [19] and Özdamar and Bozyel [20]. The second method is allowing backlog (see Zangwill [25]). Here demand must be satisfied, but the items can be produced later than requested at an extra cost. In these two cases, demand must be satisfied and the amount of lost sales for each item at each period is not given. The only information that we have are, the amount of missing capacity at each period to satisfy all demands, and the amount of backlogged items.

Usually, factories already produce at full capacity. In this case, overtimes are impossible. Meeting the demand at later period is also frequently prohibited by clients. In our case, demand should be met, but when we are in lack of capacity demand is lost at a cost instead of adding overtimes or using backlogging.

The main contributions of this paper are twofold. First, we show that the results obtained from considering relaxations based on single-period sub-model can be used to derive new valid inequalities for the MCLSSP problem. These results are derived from Miller *et al.* [17] previous work on the polyhedral structure of the single-period relaxation of the multi-item capacitated lot-sizing problem. Second, we show that these inequalities induce facets of the convex hull under certain conditions.

An outline of the remainder of the paper follows. Sections 1 and 2 describe MIP formulations of the MCLSSP problem and its single-period relaxation. We also state results concerning the generalization of the (l, S) valid inequalities. In Section 3 we present cover and reverse cover valid inequalities. In section 4 we study the polyhedral structure of the convex hull of this model and we prove that cover inequalities induce facets of the convex hull under certain conditions.

1 Formulation of the MCLSSP problem

In this section we present a MIP formulation of the MCLSSP problem, which is an extension to the classical formulation of the MCLSP problem previously studied by Trigeiro *et al.* [23] and Miller [16]. In the sequel, we consider that $i = 1, \ldots, N$ and $t = 1, \ldots, T$. We set x_{it} as the quantity of item i produced at period t. To deal with the fixed setup times and costs, we need also to define y_{it} as a binary variable equal to 1 if item i is produced at period t (i.e. if $x_{it} > 0$). The variable s_{it} is the inventory value for item i at the end of period t. The demand shortage for item i at period t is modeled by a non-negative variable r_{it} added to the production variables x_{it} with a very high unit penalty cost in the objective function, because the main goal is to satisfy the customer and thus to have the minimum amount of the requirements not met. We can notice that $r_{it} = -(s_{i,t-1} + x_{it}) + d_{it}$ if $r_{it} > 0$ and 0 otherwise.

$$\min \sum_{i=1}^{N} \sum_{t=1}^{T} \alpha_{it} x_{it} + \beta_{it} y_{it} + \gamma_{it} s_{it} + \varphi_{it} r_{it} \tag{1}$$

subject to:

$$x_{it} + r_{it} - s_{it} + s_{i,t-1} = d_{it}, \ i = 1, \ldots, N, t = 1, \ldots, T. \tag{2}$$

$$\sum_{i=1}^{N} v_{it} x_{it} + \sum_{i=1}^{N} f_{it} y_{it} \leq c_t, \ t = 1, \ldots, T. \tag{3}$$

$$x_{it} \leq \min \left\{ \frac{c_t - f_{it}}{v_{it}}, \sum_{t'=t}^{\sigma_{it}} d_{it'} \right\} y_{it}, \ i = 1, \ldots, N, t = 1, \ldots, T. \tag{4}$$

$$r_{it} \leq d_{it}, \ i = 1, \ldots, N, t = 1, \ldots, T \tag{5}$$

$$x_{it}, s_{it}, r_{it} \geq 0, \ i = 1, \ldots, N, t = 1, \ldots, T \tag{6}$$

$$y_{it} \in \{0, 1\}, \ i = 1, \ldots, N, t = 1, \ldots, T \tag{7}$$

The objective function (1) minimizes the total cost induced by the production plan (unit production costs, inventory costs, shortage costs and setup costs). Constraints (2) are the flow conservation of the inventory through the planning horizon. Constraints (3) are the capacity constraints, the overall consumption must remain lower than the available capacity. If we produce an item then the production must not exceed a maximum production level, this condition is ensured by constraints (4). Indeed, the maximum production is the minimum between the maximum quantity of the item that we can produce and the total requirement on section $[t, \ldots, \sigma_{it}]$ of the horizon (σ_{it} denote the last period at which an item i produced at period t can be consumed). Constraints (5) define upper bounds on the requirement not met for item i on period t. Constraints (6) and (7) characterize the variable's domain: x_{it}, s_{it} and r_{it} are non-negative for $i = 1, \ldots, N$ and $t = 1, \ldots, T$ and y_{it} is a binary variable for $i = 1, \ldots, N$ and $t = 1, \ldots, T$.

In the sequel of the paper, we refer to valid inequalities for the set defined by $(2) - (7)$ as valid for MCLSSP.

2 Single-period relaxation of the MCLSSP problem

Based on the previous formulation of the MCLSSP problem, we define a simplified submodel obtained by considering a single time period and the interaction with future periods. This model is called the single-period relaxation of the MCLSSP with preceding inventory. Our approach is similar that used by Constantino [7] and Miller [16] to derive a set of valid inequalities for the MCLSP problem based on a single-period relaxation. This is particularly useful to derive valid inequalities for the MCLSSP problem. In this relaxation, the production over a given period could satisfy the requirement of a section of consecutive periods. Consequently, for each period $t = 1, \ldots, T$ and each item $i = 1, \ldots, N$ we use the parameter σ_{it} previously defined with $\sigma_{it} = 1, \ldots, T$. This will enable us to create a mathematical model for each period $t = 1, \ldots, T$ which captures the interaction between the tight capacity in one hand and the requirements, the productions and the setups on the other hand from period t to σ_{it}, for each item $i = 1, \ldots, N$.
Here our goal is to derive valid inequalities for MCLSSP by considering simplified models obtained from a single-time period relaxation with preceding inventory.

Let us denote: $\delta^i_{a,b} = \sum_{t=a}^{b} d_{it}$. One simple family of valid inequalities is given by

Proposition 1. *The inequalities*

$$x_{it} + \sum_{t'=t}^{\sigma_{it}} r_{it'} + \left(s_{i,t-1} + \sum_{t'=t+1}^{\sigma_{it}} \delta^i_{t',\sigma_{it}} y_{it'} \right) \geq \delta^i_{t,\sigma_{it}}, \; i = 1, \ldots, N, t = 1, \ldots, T. \tag{8}$$

are valid for MCLSSP.

Proof. Summing the constraints (2) over the section of horizon $[t, \ldots, \sigma_{it}]$ gives:

$$\sum_{t'=t}^{\sigma_{it}} (x_{it'} + r_{it'}) - s_{i,\sigma_{it}} + s_{i,t-1} = \sum_{t'=t}^{\sigma_{it}} d_{it'}, \; i = 1, \ldots, N, t = 1, \ldots, T. \tag{9}$$

The variable x_{it} can be redefined by considering the period where the production is really consumed. This reformulation is called the facility location-based formulation introduced initially by Krarup and Bilde [11]. Therefore, we denote $w_{itt'}$ with $t' \in [t, \sigma_{it}]$ the quantity of the item i produced at period t ($t \neq 0$) and consumed at period t'. The variables w_{i0t} then represent the opening inventory of item i at the beginning of the horizon which will be consumed at period t. We will have:

$$x_{it} = \sum_{t'=t}^{T} w_{itt'},\ \ i = 1, \ldots, N, t = 1, \ldots, T. \tag{10}$$

and

$$s_{it} = \sum_{t'=0}^{t} \sum_{t''=t+1}^{T} w_{it't''},\ \ i = 1, \ldots, N, t = 1, \ldots, T. \tag{11}$$

By replacing (10) and (11) in (9), we get for each $i = 1, \ldots, N$ and $t = 1, \ldots, T$:

$$s_{i,t-1} + x_{it} + \sum_{t'=t+1}^{\sigma_{it}} \sum_{t''=t'}^{T} w_{it't''} + \sum_{t'=t}^{\sigma_{it}} r_{it'} - \sum_{t'=0}^{\sigma_{it}} \sum_{t''=\sigma_{it}+1}^{T} w_{it't''} = \sum_{t'=t}^{\sigma_{it}} d_{it'} \tag{12}$$

Moreover:

$$\sum_{t'=t+1}^{\sigma_{it}} \sum_{t''=t'}^{T} w_{it't''} = \sum_{t'=t+1}^{\sigma_{it}} \sum_{t''=t'}^{\sigma_{it}} w_{it't''} + \sum_{t'=t+1}^{\sigma_{it}} \sum_{t''=\sigma_{it}+1}^{T} w_{it't''} \tag{13}$$

and:

$$\sum_{t'=0}^{\sigma_{it}} \sum_{t''=\sigma_{it}+1}^{T} w_{it't''} = \sum_{t'=0}^{t} \sum_{t''=\sigma_{it}+1}^{T} w_{it't''} + \sum_{t'=t+1}^{\sigma_{it}} \sum_{t''=\sigma_{it}+1}^{T} w_{it't''} \tag{14}$$

By replacing (13) and (14) in (12), we get for each $i = 1, \ldots, N$ and $t = 1, \ldots, T$:

$$s_{i,t-1} + x_{it} + \sum_{t'=t+1}^{\sigma_{it}} \sum_{t''=t'}^{\sigma_{it}} w_{it't''} - \sum_{t'=0}^{t} \sum_{t''=\sigma_{it}+1}^{T} w_{it't''} + \sum_{t'=t}^{\sigma_{it}} r_{it'} = \sum_{t'=t}^{\sigma_{it}} d_{it'} \tag{15}$$

By definition of variables $w_{it't''}$, we know that:

1. $w_{it't''} \leq d_{it''} y_{it'}$
2. $w_{i0t} \leq d_{it}$
3. $\sum_{t'=0}^{t} \sum_{t''=\sigma_{it}+1}^{T} w_{it't''} = 0$

Consequently, from (15), we get for each $i = 1, \ldots, N$ and $t = 1, \ldots, T$:

$$s_{i,t-1} + x_{it} + \sum_{t'=t+1}^{\sigma_{it}} \sum_{t''=t'}^{\sigma_{it}} d_{it''} y_{it'} + \sum_{t'=t}^{\sigma_{it}} r_{it'} \geq \sum_{t'=t}^{\sigma_{it}} d_{it'}$$

Furthermore,

$$\sum_{t''=t'}^{\sigma_{it}} d_{it''} = \delta^i_{t',\sigma_{it}}$$

Finally,

$$x_{it} + \sum_{t'=t}^{\sigma_{it}} r_{it'} + s_{i,t-1} + \sum_{t'=t+1}^{\sigma_{it}} \delta^i_{t',\sigma_{it}} y_{it'} \geq \delta^i_{t,\sigma_{it}}$$

□

In the sequel of the paper, we denote by SPMCLSSP the Single-Period relaxation of the problem MCLSSP where (2) is replaced by (8). As previously mentioned, we refer to valid inequalities for the set defined by $(3) - (8)$ as valid for SPMCLSSP.

Since each period is examined separately in the SPMCLSSP, some simplifications could be done on inequations (8) . The expression $s_{i,t-1} + \sum_{t'=t+1}^{\sigma_{it}} \delta^i_{t',\sigma_{it}} y_{it'}$ can be considered as being the ending inventory of item i at period $t-1$ and denoted $\widetilde{s}_{i,t-1}$. Thus, we have $\widetilde{s}_{i,t-1} = s_{i,t-1} + \sum_{t'=t+1}^{\sigma_{it}} \delta^i_{t',\sigma_{it}} y_{it'}$. Similarly, we note $\sum_{t'=t}^{\sigma_{it}} r_{it'}$ by $\widetilde{r}_{it}$ and $\delta^i_{t,\sigma_{it}}$ by $\widetilde{d}_{it}$. The inequalities (8) are equivalent to:

$$x_{it} + \widetilde{r}_{it} + \widetilde{s}_{it} \geq \widetilde{d}_{it},\ i = 1, \ldots, N, t = 1, \ldots, T. \tag{16}$$

For the sake of simplicity, we remove the temporal index in the previous expression since we work on a single-period in SPMCLSSP.

The inequalities (8) are written:

$$x_i + \widetilde{r}_i + \widetilde{s}_i \geq \widetilde{d}_i,\ i = 1, \ldots, N. \tag{17}$$

Proposition 2. *Given a partition (U, V) of the interval $[t+1, \ldots, \sigma_{it}]$, the inequalities*

$$x_{it} + \sum_{t'=t}^{\sigma_{it}} r_{it'} + \left(s_{i,t-1} + \sum_U \delta^i_{t',\sigma_{it}} y_{it'} + \sum_V x_{it'} \right) \geq \delta^i_{t,\sigma_{it}},\ i = 1, \ldots, N, t = 1, \ldots, T \tag{18}$$

are valid for MCLSSP.

Proof. The proof is similar to the proof of proposition 1. □

The inequalities (18) are called the (l, S) inequalities for the problem SPMCLSSP.

3 Cover and reverse cover inequalities for the SPMCLSSP

In this section, we generalize some results on the cover and reverse cover inequalities defined by Miller *et al.* [17].

Definition 1. *(Cover)*
A subset of items S of $\mathcal{I}$ is known as "cover" of the problem SPMCLSSP if:

$$\lambda_S = \sum_{i \in S} \left(f_i + v_i \widetilde{d}_i \right) - c \geq 0 \tag{19}$$

For the cover S, λ_s expresses the lack of capacity when all the items of S are produced. Indeed, if $\lambda_s > 0$ then the total requirements of all the items of S are strictly higher than the available capacity.

Proposition 3. *(Cover inequalities)*

The inequality

$$\sum_{i \in S} v_i \left(\widetilde{s}_i + \widetilde{r}_i \right) \geq \lambda_S + \sum_{i \in S} \max \left\{ -f_i, v_i \widetilde{d}_i - \lambda_S \right\} (1 - y_i) \tag{20}$$

is valid for SPMCLSSP.

Proof. The proof is similar to the one presented in Miller *et al.* [17] by adding the demand shortage variables $\widetilde{r}_i$ as well as the variable resource consumption v_i.

The inequalities (17) can be written:

$$\widetilde{s}_i + \widetilde{r}_i \geq \widetilde{d}_i - x_i,\ i = 1, \ldots, N.$$

Then:

$$\sum_{i \in S} v_i \widetilde{s}_i + \sum_{i \in S} v_i \widetilde{r}_i \geq \sum_{i \in S} v_i \widetilde{d}_i - \sum_{i \in S} v_i x_i$$

If all the items of S are produced, $y_i = 1\ \forall i \in S$, from (3) we get:

$$\sum_{i \in S} v_i x_i \leq c - \sum_{i \in S} f_i$$

Then:

$$\sum_{i \in S} v_i \widetilde{s}_i + \sum_{i \in S} v_i \widetilde{r}_i \geq \sum_{i \in S} v_i \widetilde{d}_i - \left(c - \sum_{i \in S} f_i \right) = \sum_{i \in S} \left(v_i \widetilde{d}_i + f_i \right) - c$$

By replacing $\sum_{i \in S} \left(v_i \widetilde{d}_i + f_i \right) - c$ by λ_S we get:

$$\sum_{i \in S} v_i \left(\widetilde{s}_i + \widetilde{r}_i \right) \geq \lambda_S \tag{21}$$

We define a set $S^0 = \{i \in S : y_i = 0\}$ that represents the items in S that are not produced.

If $\left| S^0 \right| = 1$, we have exactly one item $i' \in S$ such that $y_{i'} = 0$.

From (21) we can write:

$$\sum_{i \in S} v_i \left(\widetilde{s}_i + \widetilde{r}_i \right) \geq \lambda_S - f_{i'} \tag{22}$$

We know that:

$$\sum_{i \in S} v_i \left(\widetilde{s}_i + \widetilde{r}_i \right) \geq v_{i'} \left(\widetilde{s}_{i'} + \widetilde{r}_{i'} \right) \geq v_{i'} \widetilde{d}_{i'} \tag{23}$$

Thus, from (22) and (23) we can conclude that:

$$\sum_{i \in S} v_i \left(\widetilde{s}_i + \widetilde{r}_i \right) \geq \lambda_S + \max \left\{ -f_{i'}, v_{i'} \widetilde{d}_{i'} - \lambda_S \right\} \tag{24}$$

Let us consider now the case where $\left| S^0 \right| > 1$. The inequality (24) can easily be generalized by considering the items in S^0 one by one. Hence, we get:

$$\sum_{i \in S} v_i \left(\widetilde{s}_i + \widetilde{r}_i \right) \geq \lambda_S + \sum_{i \in S^0} \max \left\{ -f_i, v_i \widetilde{d}_i - \lambda_S \right\} \tag{25}$$

The inequality (25) can be generalized for the set S by introducing the term $(1 - y_i)$ to take into account the production of the item. Hence, we have:

$$\sum_{i\in S} v_i\left(\widetilde{s}_i+\widetilde{r}_i\right) \geq \lambda_S + \sum_{i\in S} \max\left\{-f_i, v_i\widetilde{d}_i - \lambda_S\right\}(1-y_i)$$

□

In what follows, we describe another class of valid inequalities based on the reverse cover set.

Definition 2. *(Reverse Cover)*
A subset S of $\mathcal{I}$ is known as reverse cover of SPMCLSSP if:

$$\mu_S = c - \sum_{i\in S}\left(f_i + v_i\widetilde{d}_i\right) \geq 0 \tag{26}$$

For a reverse cover S, μ_S expresses the available capacity left when the total requirement for each item of S is produced.

Proposition 4. *Let S be a reverse cover of SPMCLSSP, $T = \mathcal{I}\setminus S$ and (T', T'') be any partition of T. The inequality*

$$\sum_{i\in S} v_i\left(\widetilde{s}_i+\widetilde{r}_i\right) \geq \left(\sum_{i\in S}\left(f_i + v_i\widetilde{d}_i\right)\right)\sum_{i\in T'} y_i - \sum_{i\in S} f_i(1-y_i) - \sum_{i\in T'}\left((c-f_i)\,y_i - v_i x_i\right) \tag{27}$$

is valid for SPMCLSSP.

Proof. The proof presented here is similar to the one described in Miller *et al.* [17]. In the following, we take into account the demand shortage variables $\widetilde{r}_i$ as well as the variable resource consumption v_i.
Let $(x^*, y^*, \widetilde{s}^*, \widetilde{r}^*)$ be any point of the convex hull of SPMCLSSP.
We have to consider three cases:
If $y_i^* = 0$ for all $i \in T'$, then the inequality is valid, because $\sum_{i\in S} v_i\left(\widetilde{s}_i^*+\widetilde{r}_i^*\right) \geq -\sum_{i\in S} f_i\left(1-\widetilde{y}_i^*\right)$.
Let $\bar{T}' = \left\{j \in T' : y_j^* = 1\right\}$

If $\left|\bar{T}'\right| = 1$, we assume that $\bar{T}' = \{i'\}$
From (3) we have:

$$c - f_{i'} \geq \sum_{i\in S}\left(v_i x_i^* + f_i y_i^*\right) + v_{i'} x_{i'}^*$$

From (17) we also have:

$$x_i^* \geq \widetilde{d}_i - \widetilde{s}_i^* - \widetilde{r}_i^*$$

Consequently, we get:

$$c - f_{i'} \geq \sum_{i\in S}\left(v_i\left(\widetilde{d}_i - \widetilde{s}_i^* - \widetilde{r}_i^*\right) + f_i y_i^*\right) + v_{i'} x_{i'}^*$$

Which gives:

$$\sum_{i\in S} v_i\left(\widetilde{s}_i^*+\widetilde{r}_i^*\right) \geq \sum_{i\in S} v_i\widetilde{d}_i + \sum_{i\in S} f_i y_i^* - \left((c-f_{i'}) - v_{i'} x_{i'}^*\right)$$

The inequality

$$\sum_{i\in S} v_i\left(\widetilde{s}_i^* + \widetilde{r}_i^*\right) \geq \left(\sum_{i\in S}\left(f_i + v_i\widetilde{d}_i\right)\right) - \sum_{i\in S} f_i\left(1-y_i^*\right) - \left(\left(c - f_{i'}\right) - v_{i'}x_{i'}^*\right) \tag{28}$$

is thus valid for SPMCLSSP.
If $\left|\bar{T}'\right| > 1$, the inequality (28) can be easily generalized by considering the items of $\bar{T}'$ one by one. The inequality (27) follows.

□

4 Polyhedral results

In this section we present some results concerning the polyhedral structure of SPMCLSSP. We will discuss the extreme points and rays of conv(SPMCLSSP), conv(SPMCLSSP) denote the convex hull of SPMCLSSP. We show that cover inqualities are facet defining of conv(SPMCLSPS).

4.1 Extreme points and rays

Given an extreme point $(\bar{x}, \bar{y}, \bar{s}, \bar{r})$ of conv(SPMCLSSP). Let, $Q = \{i \in \mathcal{I} : \bar{y}_i = 1\}$. Also, let $Q_u = \left\{i \in Q : \bar{x}_i = \widetilde{d}_i\right\}$, $Q_m = \left\{i \in Q : \bar{x}_i > 0, \bar{x}_i \neq \widetilde{d}_i\right\}$, $Q_l = \{i \in Q : \bar{x}_i = 0\}$. For a given i we define types of $(\bar{x}_i, \bar{y}_i, \bar{s}_i, \bar{r}_i)$

Type 1 : $\bar{x}_i = \widetilde{d}_i, \bar{y}_i = 1, \bar{s}_i = 0, \bar{r}_i = 0$.
Type 2 : $\bar{x}_i = 0, \bar{y}_i = 1, \bar{s}_i = \widetilde{d}_i, \bar{r}_i = 0$.
Type 3 : $\bar{x}_i = 0, \bar{y}_i = 1, \bar{s}_i = 0, \bar{r}_i = \widetilde{d}_i$.
Type 4 : $\bar{x}_i = 0, \bar{y}_i = 0, \bar{s}_i = \widetilde{d}_i, \bar{r}_i = 0$.
Type 5 : $\bar{x}_i = 0, \bar{y}_i = 0, \bar{s}_i = 0, \bar{r}_i = \widetilde{d}_i$.

Proposition 5. *In every extreme point $(\bar{x}, \bar{y}, \bar{s}, \bar{r})$ of conv(SPMCLSSP), $(\bar{x}_i, \bar{y}_i, \bar{s}_i, \bar{r}_i)$ is of type 1, $i \in Q_u$; $(\bar{x}_i, \bar{y}_i, \bar{s}_i, \bar{r}_i)$ is of type 2 or type 3, $i \in Q_l$; and $(\bar{x}_i, \bar{y}_i, \bar{s}_i, \bar{r}_i)$ is of type 4 or type 5, $i \in N \setminus Q$. Moreover, either $Q_m = \emptyset$ or $|Q_m| = 1$ and*

$$\bar{x}_m = c - \sum_{i\in Q_u}\left(f_i + v_i\widetilde{d}_i\right) - \sum_{i\in Q_l} f_i - f_m,\ \bar{y}_m = 1,\ \bar{s}_m = (\widetilde{d}_i - \bar{x}_m)^+,\ \bar{r}_m = 0.$$

Or

$$\bar{x}_m = c - \sum_{i\in Q_u}\left(f_i + v_i\widetilde{d}_i\right) - \sum_{i\in Q_l} f_i - f_m,\ \bar{y}_m = 1,\ \bar{s}_m = 0,\ \bar{r}_m = (\widetilde{d}_i - \bar{x}_m)^+.$$

Proof. The proposition says that there is only one $i \in \mathcal{I}$ for which the constraint (20) plays a part in determining the value $\bar{x}_i$. For the other $i \in \mathcal{I}$, x_i is determined by the other constraints. If $|Q_m| > 1$ for some $(\bar{x}, \bar{y}, \bar{s}, \bar{r})$ in conv(SPMCLSSP), then it's not difficult to construct two points such that $(\bar{x}, \bar{y}, \bar{s}, \bar{r})$ is a convex combination of the two.

□

We will denote SPMCLSSP^0 the set of points defined by constraints (29)-(33).

$$x_i + s_i + r_i \geq 0,\ i = 1, \ldots, N. \tag{29}$$

$$\sum_{i=1}^{N} v_i x_i + \sum_{i=1}^{N} f_i y_i \leq 0 \tag{30}$$

$$x_i \leq \min\left\{\frac{c - f_i}{v_i}, \widetilde{d}_i\right\} y_{it},\ i = 1, \ldots, N \tag{31}$$

$$x_i, s_i, r_i \geq 0,\ i = 1, \ldots, N. \tag{32}$$

$$y_i \in \{0, 1\},\ i = 1, \ldots, N. \tag{33}$$

Definition 3. *$(\hat{x}, \hat{y}, \hat{s}, \hat{r})$ is a ray of conv(SPMCLSSP) if $(\hat{x}, \hat{y}, \hat{s}, \hat{r}) \in SPMCLSSP^0 \setminus \{0\}$.*

Definition 4. *a ray $(\hat{x}, \hat{y}, \hat{s}, \hat{r})$ of conv(SPMCLSSP) is an extreme ray if there does no exist rays $(\hat{x}^1, \hat{y}^1, \hat{s}^1, \hat{r}^1), (\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2) \in SPMCLSSP^0 \backslash \{0\}$, $(\hat{x}^1, \hat{y}^1, \hat{s}^1, \hat{r}^1) \neq \eta(\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2)$ for any $\eta \in \mathbb{R}_+$, sush that : $(\hat{x}, \hat{y}, \hat{s}, \hat{r}) = \frac{1}{2}(\hat{x}^1, \hat{y}^1, \hat{s}^1, \hat{r}^1) + \frac{1}{2}(\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2)$.*

Proposition 6. *All extreme rays of conv(SPMCLSSP) have the form $x_i = y_i = 0, s_i = 1, r_i = 0$ or $x_i = y_i = 0, s_i = 0, r_i = 1$ for some $i \in \mathcal{I}$ and $x_j = y_j = s_j = r_j = 0,\ j \neq i$.*

Proof. From (30), (32) and (33) we have : $\hat{x}_i = \hat{y}_i = 0$ for all $i \in \mathcal{I}$. Then extreme rays of conv(SPMCLSSP) have the form : $(0, 0, \hat{s}, \hat{r})$.

if for a ray $(0, 0, \hat{s}, \hat{r})$ we have more than one nonzero components, then it's not difficult to construct two rays such that $(0, 0, \hat{s}, \hat{r})$ is a convex combination of the two.

Rays that have the form $x_i = y_i = 0$, $s_i = \hat{s}_i$, $r_i = 0$ or $x_i = y_i = 0$, $s_i = 0$, $r_i = \hat{r}_i$ for some $i \in \mathcal{I}$ and $x_j = y_j = s_j = r_j = 0$, $j \neq i$ connot be expressed as a convex combination of two rays.

All the extreme rays of conv(SPMCLSSP) have the form $x_i = y_i = 0$, $s_i = 1$, $r_i = 0$ or $x_i = y_i = 0$, $s_i = 0$, $r_i = 1$ for some $i \in \mathcal{I}$ and $x_j = y_j = s_j = r_j = 0$, $j \neq i$. ☐

Proposition 7. *Given a cover S of SPMCLSPS, and an order of items $i \in S$ such that $f_{[1]} + v_{[1]}\widetilde{d}_{[1]} \geq \cdots \geq f_{[|S|]} + v_{[|S|]}\widetilde{d}_{[|S|]}$. Let $T = \mathcal{I} \setminus S$. We define $\mu_1 = f_{[1]} + v_{[1]}\widetilde{d}_{[1]} - \lambda_S$. If $|S| \geq 2$, $\lambda_S > 0$, $f_{[2]} + v_{[2]}\widetilde{d}_{[2]} \geq \lambda_S$ and $f_i < \mu_1$, $i \in T$. The inequality (20) induces a facet of conv(SPMCLSPS).*

Proof. conv(SPMCLSPS) is full-dimentional. It suffices to show that the intersection of the hyperplane defined by (20) and conv(SPMCLSPS) has dimension $4N - 1$. Note that the $2|T|$ rays with $s_i = 1$ and $r_i = 1$, $i \in T$, are extreme rays of conv(SPMCLSPS) and also lie in the hyperplan defined by (20). We will take into account only the T extreme rays $s_i = 1$, $i \in T$. Therefore, it suffices to show $4P - |T|$ linearly independant feasible points that lies in this hyperplane, and that are also linearly independant of the $|T|$ extreme rays $s_i = 1$. Consider the $4P - |T|$ points.

- For each $i' \in S \setminus [1]$,
  $x_{i'} = (\frac{v_{i'}\widetilde{d}_{i'} - \lambda_S}{v_{i'}})^+, y_{i'} = 1, s_{i'} = \widetilde{d}_{i'} - x_{i'}, r_{i'} = 0$
  $x_{[1]} = \widetilde{d}_{[1]} - (\frac{\lambda_S - v_{i'}\widetilde{d}_{i'}}{v_{[1]}})^+, y_{[1]} = 1, s_{[1]} = \widetilde{d}_{[1]} - x_{[1]}, r_{[1]} = 0$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{i' \cup [1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S$
  $|S|$ - 1 points.
- For [1],
  $x_{[1]} = (\frac{v_{[1]}\widetilde{d}_{[1]} - \lambda_S}{v_{[1]}})^+, y_{[1]} = 1, s_{[1]} = \widetilde{d}_{[1]} - x_{[1]}, r_{[1]} = 0$

$x_{[2]} = \widetilde{d}_{[2]} - (\frac{\lambda_S - v_{[1]}\widetilde{d}_{[1]}}{v_{[2]}})^+, y_{[2]} = 1, s_{[2]} = \widetilde{d}_{[2]} - x_{[2]}, r_{[2]} = 0$
$x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{[2] \cup [1]\}$
$x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S$
1 points.

- For each $i' \in S \setminus [1]$,
  $x_{i'} = y_{i'} = 0, s_{i'} = \widetilde{d}_{i'}, r_{i'} = 0$
  $x_{[1]} = \widetilde{d}_{[1]} - (\frac{\lambda_S - f_{i'} - v_{i'}\widetilde{d}_{i'}}{v_{[1]}})^+, y_{[1]} = 1, s_{[1]} = \widetilde{d}_{[1]} - x_{[1]}, r_{[1]} = 0$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{i' \cup [1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S$
  $|S|$ - 1 points.
- For [1],
  $x_{[1]} = y_{[1]} = 0, s_{[1]} = \widetilde{d}_{[1]}, r_{[1]} = 0$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{[1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S$
  1 points.
- For each $i' \in S \setminus [1]$,
  $x_{i'} = \widetilde{d}_{i'} + (\frac{v_{[1]}\widetilde{d}_{[1]} + f_{[1]} - \lambda_S}{v_{i'}})^+, y_{i'} = 1, s_{i'} = 0, r_{i'} = 0$
  $x_{[1]} = y_{[1]} = 0, s_{[1]} = \widetilde{d}_{[1]}, r_{[1]} = 0$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{i' \cup [1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S$
  $|S|$ - 1 points.
- For [1],
  $x_{[1]} = \widetilde{d}_{[1]} + (\frac{v_{[2]}\widetilde{d}_{[2]} + f_{[2]} - \lambda_S}{v_{[1]}})^+, y_{[1]} = 1, s_{[1]} = 0, r_{[1]} = 0$
  $x_{[2]} = y_{[2]} = 0, s_{[2]} = \widetilde{d}_{[2]}, r_{[2]} = 0$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{[2] \cup [1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S$
  1 points.
- For each $i' \in S \setminus [1]$,
  $x_{i'} = \widetilde{d}_{i'} + (\frac{v_{[1]}\widetilde{d}_{[1]} + f_{[1]} - \lambda_S}{v_{i'}})^+, y_{i'} = 1, s_{i'} = 0, r_{i'} = 0$
  $x_{[1]} = y_{[1]} = 0, s_{[1]} = 0, r_{[1]} = \widetilde{d}_{[1]}$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{i' \cup [1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S$
  $|S|$ - 1 points.
- For [1],
  $x_{[1]} = \widetilde{d}_{[1]} + (\frac{v_{[2]}\widetilde{d}_{[2]} + f_{[2]} - \lambda_S}{v_{[1]}})^+, y_{[1]} = 1, s_{[1]} = 0, r_{[1]} = 0$
  $x_{[2]} = y_{[2]} = 0, s_{[2]} = 0, r_{[2]} = \widetilde{d}_{[2]}$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{[2] \cup [1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S$
  1 points.
- For each $i' \in T$,
  $x_{i'} = 0, y_{i'} = 1, s_{i'} = \widetilde{d}_{i'}, r_{i'} = 0$
  $x_{[1]} = y_{[1]} = 0, s_{[1]} = \widetilde{d}_{[1]}, r_{[1]} = 0$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{[1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S \cup \{i'\}$
  $|S|$ - 1 points.

- For each $i' \in T$,
  $x_{i'} = (\frac{v_{[1]}\widetilde{d}_{[1]}+f_{[1]}-\lambda_S - f_{i'}}{v_{i'}})^+, y_{i'} = 1, s_{i'} = \widetilde{d}_{i'} - x_{i'}, r_{i'} = 0$
  $x_{[1]} = y_{[1]} = 0, s_{[1]} = \widetilde{d}_{[1]}, r_{[1]} = 0$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{[1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S \cup \{i'\}$
  $|S|$ - 1 points.
- For each $i' \in T$,
  $x_{i'} = (\frac{v_{[1]}\widetilde{d}_{[1]}+f_{[1]}-\lambda_S - f_{i'}}{v_{i'}})^+, y_{i'} = 1, s_{i'} = \widetilde{d}_{i'} - x_{i'}, r_{i'} = 0$
  $x_{[1]} = y_{[1]} = 0, s_{[1]} = 0, r_{[1]} = \widetilde{d}_{[1]}$
  $x_i = \widetilde{d}_i, y_i = 1, s_i = 0, r_i = 0,\ i \in S \setminus \{[1]\}$
  $x_i = y_i = 0, s_i = \widetilde{d}_i, r_i = 0,\ i \notin S \cup \{i'\}$
  $|S|$ - 1 points.

We have showed $(4\,|S| + 3\,|T|) = (4N - |T|)$ independant points. It can be checked that these $(4N - |T|)$ points and the $|T|$ rays $s_i = 1$, $i \in T$, are linearly independant.

□

5 Conclusion

We proposed a mathematical formulation of a capacitated lot-sizing problem with additional constraints. A polyhedral approach has yielded strong valid inequalities. We also study the polyhedral structure of the convex hull of the signle-period relaxation of this model. We prove that these inequalities induce facets of the convex hull under certain conditions. By following the same approach, we can prove that reverse cover inequalities are also facet defining under certain conditions. The valid inequalities presented in this document were generalized to take into account other constraints that occur frequently in industrial situations. Some extensions could be done when we have to deal with setup constraints on groups of items or minimal production and minimum run constraints. These inequalities were also generalized when more than one resource is available. Further research will be devoted to find new inequalities to take into account several production processes. It would be also interesting to use this approach in conjunction with a time decomposition heuristic.

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