How to target ellipse in general form through 5 points

We've got five points P_1, P_2, P_3, P_4 and P_5. To find exact equation of general ellipse getting through these points we have to calculate parameters A, B, C, D, E and F of general formula of ellipse: $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. Having five points with coordinates x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5 we fill up matrices to find set of parameters solving equations by determinant method:

$$\begin{cases} Ax_1^2+Bx_1y_1+Cy_1^2+Dx_1+Ey_1+F=0 \\ Ax_2^2+Bx_2y_2+Cy_2^2+Dx_2+Ey_2+F=0 \\ Ax_3^2+Bx_3y_3+Cy_3^2+Dx_3+Ey_3+F=0 \\ Ax_4^2+Bx_4y_4+Cy_4^2+Dx_4+Ey_4+F=0 \\ Ax_5^2+Bx_5y_5+Cy_5^2+Dx_5+Ey_5+F=0 \end{cases}$$

$$W=\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 \\ x_2^2 & x_2y_1 & y_2^2 & x_2 & y_2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 \end{bmatrix} \quad W_A=\begin{bmatrix} 1 & x_1y_1 & y_1^2 & x_1 & y_1 \\ 1 & x_2y_1 & y_2^2 & x_2 & y_2 \\ 1 & x_3y_3 & y_3^2 & x_3 & y_3 \\ 1 & x_4y_4 & y_4^2 & x_4 & y_4 \\ 1 & x_5y_5 & y_5^2 & x_5 & y_5 \end{bmatrix} \quad W_B=\begin{bmatrix} x_1^2 & 1 & y_1^2 & x_1 & y_1 \\ x_2^2 & 1 & y_2^2 & x_2 & y_2 \\ x_3^2 & 1 & y_3^2 & x_3 & y_3 \\ x_4^2 & 1 & y_4^2 & x_4 & y_4 \\ x_5^2 & 1 & y_5^2 & x_5 & y_5 \end{bmatrix}$$

$$W_C=\begin{bmatrix} x_1^2 & x_1y_1 & 1 & x_1 & y_1 \\ x_2^2 & x_2y_1 & 1 & x_2 & y_2 \\ x_3^2 & x_3y_3 & 1 & x_3 & y_3 \\ x_4^2 & x_4y_4 & 1 & x_4 & y_4 \\ x_5^2 & x_5y_5 & 1 & x_5 & y_5 \end{bmatrix} \quad W_D=\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & 1 & y_1 \\ x_2^2 & x_2y_1 & y_2^2 & 1 & y_2 \\ x_3^2 & x_3y_3 & y_3^2 & 1 & y_3 \\ x_4^2 & x_4y_4 & y_4^2 & 1 & y_4 \\ x_5^2 & x_5y_5 & y_5^2 & 1 & y_5 \end{bmatrix} \quad W_E=\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & 1 \\ x_2^2 & x_2y_1 & y_2^2 & x_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & 1 \end{bmatrix}$$

After calculating determinants for all six matrices, we get parameters A, B, C, D and E dividing relevant determinants by main determinant W: $A=W_A/W$, $B=W_B/W$, $C=W_C/W$, $D=W_D/W$ and $E=W_E/W$.

After that we calculate F parameter of one of equations, i. e. the first one:

$$F=-Ax_1^2-Bx_1y_1-Cy_1^2-Dx_1-Ey_1$$

It always gives -1.

Having full set of parameters of ellipse we use following formulas for calculating centre points x_0 and y_0, height and width of ellipse – *a* and *b*, and the angle of rotation of an ellipse: θ.

$$\Delta=B^2-4AC$$

$$x_0=(2CD-BE)/\Delta$$

$$y_0=(2AE-BD)/\Delta$$

$$a=\frac{-\sqrt{2(AE^2+CD^2-BDE+\Delta\cdot F)(A+C+\sqrt{(A-C)^2+B^2})}}{\Delta}$$

$$b=\frac{-\sqrt{2(AE^2+CD^2-BDE+\Delta\cdot F)(A+C-\sqrt{(A-C)^2+B^2})}}{\Delta}$$

$$Y=-B$$

$$X=C-A$$

$$\theta=\begin{cases} 0.5\,Atan(Y/X), & X>0\wedge Y\geq 0\\ 0.5(\pi-Atan(|(Y/X)|)), & X<0\wedge Y\geq 0\\ 0.5(\pi+Atan(Y/X)), & X<0\wedge Y<0\\ 0.5(2\pi-Atan(|(Y/X)|)), & X>0\wedge Y<0\\ \frac{\pi}{4}, & X=0\wedge Y>0\\ \frac{-\pi}{4}, & X=0\wedge Y<0\\ 0, & X=0\wedge Y=0 \end{cases}$$

Parametric equation of ellipse with width and height a and b is:

$$X(c) = a\,\mathrm{Sin}(2\pi c)$$
$$Y(c) = b\,\mathrm{Cos}(2\pi c)$$

If point P(X(c);Y(c)) is rotated by angle θ, we transform it's coordinates by multiplication of it by the matrix of rotation along Z axis:

$$\begin{bmatrix} X\\ Y\end{bmatrix}=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\times\begin{bmatrix}X(c)\\ Y(c)\end{bmatrix}$$

That after multiplication of the matrix by the vector gives two equations:

$$X=X(c)\,\mathrm{Cos}(\theta) - Y(c)\,\mathrm{Sin}(\theta)$$
$$Y=X(c)\,\mathrm{Sin}(\theta) + Y(c)\,\mathrm{Cos}(\theta)$$

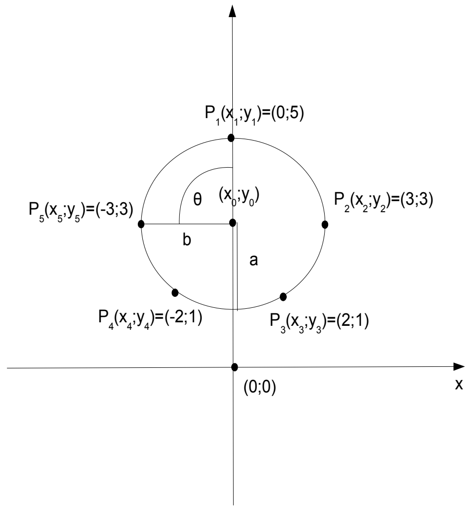
After substituting parametric equation of ellipse :

$$X(c) = a\,\mathrm{Sin}(2\pi c)\,\mathrm{Cos}(\theta) - b\,\mathrm{Cos}(2\pi c)\,\mathrm{Sin}(\theta)$$
$$Y(c) = a\,\mathrm{Sin}(2\pi c)\,\mathrm{Sin}(\theta) + b\,\mathrm{Cos}(2\pi c)\,\mathrm{Cos}(\theta)$$

After setting the ellipse in points x_0 and y_0 by translation, we get parametric equations of general form of ellipse, written over five given points, where c ranges from 0 to 1 returning coordinates of a point P(X(c),Y(c)) moving along all the length of an ellipse:

$$X(c) = x_0 + a\,\mathrm{Sin}(2\pi c)\,\mathrm{Cos}(\theta) - b\,\mathrm{Cos}(2\pi c)\,\mathrm{Sin}(\theta)$$
$$Y(c) = y_0 + a\,\mathrm{Sin}(2\pi c)\,\mathrm{Sin}(\theta) + b\,\mathrm{Cos}(2\pi c)\,\mathrm{Cos}(\theta)$$

Example:



There are five points: $P_1(0;5)$, $P_2(3;3)$, $P_3(2;1)$, $P_4(-2;1)$ and $P_5(-3;3)$. For these points we fill up a matrix of main determinant:

$$W = \begin{bmatrix} 0 & 0 & 25 & 0 & 5 \\ 9 & 9 & 9 & 3 & 3 \\ 4 & 2 & 1 & 2 & 1 \\ 4 & -2 & 1 & -2 & 1 \\ 9 & -9 & 9 & -3 & 3 \end{bmatrix}$$

Another matrices are made of the main one with filling up each column with ones, for each determinant respectively, like in equations above.

After calculations, we get main determinant: W=-2880 and following set of determinants: W_A=768, W_B=0, W_C=1344, W_D=1344, W_E=-7296. Dividing each of these by the main one, we get set of parameters of the ellipse: A=-0,2(6), B=0, C=-0,4(6), D=0, E=2,5(3). Substituting to main equation of ellipse and transforming for F, we get:

$$F = 0{,}2(6)x^2 - 0xy + 0{,}4(6)y^2 - 0x - 2{,}5(3)y$$

Calculating this formula for the first point we get:

$$F = 0{,}2(6) \cdot 0^2 - 0 \cdot 0 \cdot 5 + 0{,}4(6) \cdot 5^2 - 0 \cdot 0 - 2{,}5(3) \cdot 5 =$$
$$= 0 - 0 + 11{,}(6) - 0 - 12{,}(6) =$$
$$= -1$$

So, the equation of general ellipse getting through our particular points P_1 to P_5 is:

$$-0{,}2(6)x^2 - 0{,}4(6)y^2 + 2{,}5(3)y - 1 = 0$$

Further, we calculate values necessary for drawing the ellipse:

$\Delta = 0^2 - 4 \cdot -0{,}2(6) \cdot -0{,}4(6) = -0{,}49(7)$

$x_0 = (2 \cdot -0{,}4(6) \cdot 0 - 0 \cdot 2{,}5(3)) \,/ -0{,}49(7) = 0$

$y_0 = (2 \cdot -0{,}2(6) \cdot 2{,}5(3) - 0 \cdot 0) \,/ -0{,}49(7) = 2{,}(714285)$

$$a = \frac{-\sqrt{2(-0{,}2(6) \cdot 2{,}5(3)^2 - 0{,}4(6) \cdot 0^2 - 0 \cdot 0 \cdot 2{,}5(3) - 0{,}49(7) \cdot (-1))(-0{,}2(6) - 0{,}4(6) + \sqrt{(-0{,}2(6) - 0{,}4(6))^2 + 0^2})}}{-0{,}49(7)} = 2{,}2857$$

$$b = \frac{-\sqrt{2(-0{,}2(6) \cdot 2{,}5(3)^2 - 0{,}4(6) \cdot 0^2 - 0 \cdot 0 \cdot 2{,}5(3) - 0{,}49(7) \cdot (-1))(-0{,}2(6) - 0{,}4(6) - \sqrt{(-0{,}2(6) - 0{,}4(6))^2 + 0^2})}}{-0{,}49(7)} = 3{,}0257$$

$Y = 0$

$X = -0{,}4(6) + 0{,}2(6) = -0{,}2$

$\theta = 0{,}5 \cdot (\pi - \mathrm{Atan}(|0/-0{,}2|)) = 0{,}5 \cdot (\pi - 0) = \frac{\pi}{2}$

Finally our parametric equation of general ellipse getting through our three given points is:

$$X(c) = 0 + (2{,}2857 \,\mathrm{Sin}(2\pi c) \,\mathrm{Cos}(\tfrac{\pi}{2}) - 3{,}0257 \,\mathrm{Cos}(2\pi c) \,\mathrm{Sin}(\tfrac{\pi}{2}))$$
$$Y(c) = 2{,}(714285) + (2{,}2857 \,\mathrm{Sin}(2\pi c) \,\mathrm{Sin}(\tfrac{\pi}{2}) + 3{,}0257 \,\mathrm{Cos}(2\pi c) \,\mathrm{Cos}(\tfrac{\pi}{2}))$$

Which, after reducing values of trigonometric functions is:

$$X(c) = -3{,}0257 \,\mathrm{Cos}(2\pi c)$$
$$Y(c) = 2{,}(714285) + 2{,}2857 \,\mathrm{Sin}(2\pi c)$$